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LETTER TO THE EDITOR

Number of energy levels outside Wigner's semicircle

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Abstract. The total probability outside Wigner's semicircle is calculated using Mehler's formula for the Hermite polynomials. In the limit $N \to \infty$, N being the dimension of the random Gaussian Hermitian matrix, it is shown that this probability is $0.05N^{-3/2}$. Some implications of this small probability are discussed.

Since the introduction of random matrix ensembles by Wigner, one of the most interesting problems has been to study the probability density function of a single eigenvalue (Mehta 1967). It was soon established that this distribution is approximately a semicircle. Further interest in the problem arose when Bronk (1964) studied the number of energy levels outside the semicircle in the hope that there may still be a large number of them in the tail of the distribution, thus representing the actual behaviour of the nuclear energy level density in that region. More recently it has been pointed out (Edwards and Jones 1976) that even if there are a few such levels they may be interpreted as a collective eigenvalue of the many-body system.

As has been realised in the past, the problem is enormously complicated (Mehta 1967, Bronk 1964) and no simple expression can be derived for the total number of energy levels outside the semicircle in the limit of large N, N being the dimension of the random matrix.

The purpose of the present note is to give a derivation based on Mehler's formula (Bateman 1953) for the Hermite polynomials which gives this probability. In the derivation we shall consider only a Gaussian unitary ensemble for which the probability density function of a single eigenvalue has a simple form. It is given by (Mehta 1967)

$$P(x)=\frac{1}{N}\sum_{m=0}^{N-1}\phi_m^2(x), \tag{1}$$

where ϕ_m are normalised harmonic oscillator wavefunctions. Using Mehler's formula (Bateman 1953) for the squares of the Hermite polynomials, the total probability p outside the semicircle is given by

$$p=\frac{2}{\sqrt{\pi}}\frac{1}{N}\frac{1}{2\pi \mathrm{i}}\oint\sum_{m=0}^{N-1}\frac{\mathrm{d}z}{z^{m+1}}\frac{1}{(1-z^2)^{1/2}}\int_{\sqrt{2N}}^{\infty}\mathrm{d}x\,\exp\left(-x^2\frac{1-z}{1+z}\right), \tag{2}$$

where the contour is a circle of radius less than unity.

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Using the expansion of the complementary error function for large N and a few other simplifications we obtain

$$p=\frac{\sqrt{2}}{3\pi}\frac{\exp(-2N)}{N^{3/2}}\frac{\Gamma(N+\frac{3}{2})}{\Gamma(N)}\sum_r\frac{(2N)^r}{r!}$$
$$\times[(1-N)_r/(-\tfrac{1}{2}-N)_r]F(-N,-\tfrac{3}{2};-\tfrac{1}{2}-N+r;-1), \quad (3)$$

where F is the hypergeometric function and $(Z)_n$ is Pochhammer's symbol (Abramowitz and Stegun 1965).

For large values of N we can write F as

$$F(-N,b;c;-1)=\sum_{n=0}^{m}\frac{(b)_nN^n}{(c)_nn!} \quad (4a)$$
$$=\frac{1}{2\pi \mathrm{i}}\oint\sum_{r=0}^{m}\frac{\mathrm{d}z}{z^{r+1}}M(b,c,Nz), \quad (4b)$$

where M is the confluent hypergeometric function (Abramowitz and Stegun 1965) and m is an integer $\ll N$ but $\gg 1$ and the contour is a circle of radius greater than unity. Using a formula which gives sums over confluent hypergeometric functions (Hansen 1975) and after some simplifications we obtain

$$p=\frac{\sqrt{2}}{3\pi}\frac{1}{N^{3/2}}\frac{\Gamma(N+\frac{3}{2})}{\Gamma(N)}\frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}z}{z^{r+1}}M(-\tfrac{3}{2},-\tfrac{1}{2}-N,N(z-2)). \quad (5)$$

Writing $\Sigma_{r=0}^{m}(1/z^{r+1})$ as $(z-1)^{-1}-\Sigma_{r=m+1}^{\infty}(1/z^{r+1})$ we obtain from expression (5), for large values of m,

$$p=\frac{\sqrt{2}}{3\pi}\frac{1}{N^{3/2}}\frac{\Gamma(N+\frac{3}{2})}{\Gamma(N)}M(-\tfrac{3}{2},-\tfrac{1}{2}-N,-N). \quad (6)$$

Finally using the asymptotic expansion (Bateman 1953) of the confluent hypergeometric function M and the gamma function, we obtain the following value of p in the limit $N\to\infty$,

$$p=0.05/N^{3/2}. \quad (7)$$

The first remark which we would like to make is that one can derive the semicircular distribution itself using the above formulation. This can be seen by noting that the confluent hypergeometric function in expression (6) is replaced by $M(-\frac{1}{2},-N+\frac{1}{2},N-x^2)$. If $|N-x^2|/|-N+\frac{1}{2}|<1$ then the asymptotic form (Bateman 1953) of M gives $P(x)=\text{constant}[(4N-1)-2x^2]^{1/2}$. This provides a very nice check on the present formulation.

The second remark is that the total number of eigenvalues outside Wigner's semicircle is given by $Np=0.05/\sqrt{N}$. This implies that one cannot obtain more than a minute fraction of a single energy level outside the semicircle. Thus there are no eigenvalues outside the semicircle which could have been interpreted as a collective mode. The smallness of Np is also evident from the plots of Np given in an unpublished paper of Wigner (1962).

The last remark is that since p is less than $N^{-3/2}$ any corrections of the order $1/N$ can be included (Ullah 1981) in the semicircle itself.

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